

Boundary values as Hamiltonian variables. II. Graded structures

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Abstract

It is shown that new Poisson brackets proposed in Part I of this work (hep-th/9305133) arise naturally in an extension of the formal variational calculus incorporating divergences. The linear spaces of local functionals, evolutionary vector fields, functional forms, multi-vectors and differential operators become graded with respect to divergences. Bilinear operations, such as action of vector fields onto functionals, commutator of vector fields, interior product of forms and vectors and the Schouten-Nijenhuis bracket are compatible with the grading. A definition of the adjoint graded operator is proposed and skew-adjoint operators are constructed with the help of boundary terms. Fulfilment of the Jacobi identity for the new Poisson brackets is shown to be equivalent to the vanishing of the Schouten-Nijenhuis bracket for Poisson bivector with itself. The simple procedure for testing this condition proposed by P. Olver is applicable with a minimal modification. It is demonstrated that the second structure of the Korteweg-de Vries equation is not Hamiltonian with respect to the new brackets.

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1 Introduction

In the first part of this work[1] (below simply I) field theory Poisson brackets have been constructed which fulfil the Jacobi identity under arbitrary boundary conditions. The purpose of this paper is to show that these brackets replace the standard ones in the extension of the formal variational calculus onto total divergences. This extension consists in the introduction of a new grading for linear spaces of local functionals, vector fields, functional forms, multi-vectors and differential operators. Nonstandard Rules 4.2 and 5.4 of I for the treatment of distributions on closed domains can be better understood in this context. The former can now be treated as a definition of the interior product of 1-forms and multi-vectors or, vice versa, 1-vectors and differential forms, and the latter as a manifestation of compatibility of the bilinear operations with the grading.

We show that the Jacobi identity for the new Poisson brackets can be verified without lengthy calculations of I. Its fulfilment is equivalent to the vanishing of the Schouten–Nijenhuis bracket of the Poisson bivector with itself. And in its turn this condition can be easily tested along with the procedure proposed by Olver[2] with a minimal modification of it. So, more attention than in I is paid here to nonultralocal Hamiltonian operators with nonconstant coefficients. It occurs that not all operators which are Hamiltonian with respect to the standard brackets remain Hamiltonian in relation to the new brackets. For example, the second structure of the Korteweg–de Vries equation is not Hamiltonian.

As a rule, we use the same notations as in I, except that we change the notation for the Fréchet derivative from D_f to f' . We still find convenient to represent integrals over finite domain Ω through integrals over the infinite space \mathbb{R}^n by inserting into all integrands the characteristic function θ_Ω . Then the formalism seems closer to the standard formal variational calculus where local functionals and functional forms are defined modulo divergences. But the formal divergences that we discard here are integrated to zero under arbitrary conditions on the boundary of the finite domain, whereas real divergences are incorporated into graded structures. All the operations introduced below are compatible both with discarding formal total divergences (if one object is a formal divergence than the result of operation is also formal divergence) and with grading (i.e., the same is valid for real divergences). Extension of the space of differential operators by admitting their grading permits to use the concept of adjoint operator. So, skew-adjoint operators can now be constructed and the Poisson bracket formulas become shorter, though their content is the same. Nevertheless, in the proof of the Jacobi identity we prefer to use the old notations for easier comparison with partial proofs of I.

The content of the paper is as follows. Sections 2 and 3 are devoted to the introduction of the grading for the basic structures of the formal variational calculus: local functionals, vector fields and differential forms. Section 4 deals with graded differential operators and their adjoints. In Section 5 we discuss multi-vectors and the Schouten–Nijenhuis bracket. Section 6 is devoted to concepts of the Hamiltonian formalism with the proof of Jacobi identity postponed until Section 7. At last, in Section 8 we consider two examples: the second structure of Korteweg–de Vries equation and the 2-dimensional flow of the ideal fluid. A short summary is given in Conclusion.

2 Local functionals and evolutionary vector fields

Let us start with notions from the theory of graded spaces as they are given in Ref. [3]. A *grading* in linear space L is a decomposition of it into direct sum of subspaces, with a special value of some function p (grading function) assigned to all the elements of any subspace.

Below the function p takes its values in the set of all positive multi-indices $J = (j_1, \dots, j_n)$ and so,

$$L = \bigoplus_{J=0}^{\infty} L^{(J)}.$$

Elements of each subspace are called homogeneous.

A bilinear operation $x, y \mapsto x \circ y$, defined on L , is said to be *compatible with the grading* if the product of any homogeneous elements is also homogeneous, and if

$$p(x \circ y) = p(x) + p(y).$$

Now turn to concrete structures. The space of local functionals \mathcal{A} has already been defined in I. Here we will call the expression given in Definition 2.1 of I the *canonical form of a local functional*. We formally extend that definition by allowing local functionals to be written as follows

$$F = \sum_{J=0}^{\infty} \int D_J \theta_{\Omega} f^{(J)}(\phi_A(x), D_K \phi_A(x)) d^n x = \sum \int \theta^{(J)} f^{(J)},$$

where in accordance with the previous definition only a finite number of terms is allowed. Here and below we simplify the notation for derivatives of θ and remove Ω . Of course, any functional of such a form can be transformed to the form used in I through integration by parts

$$F = \int \theta_{\Omega} f = \int_{\Omega} f,$$

where

$$f = \sum (-1)^{|J|} D_J f^{(J)}.$$

So, the formal integration by parts over infinite space \mathbb{R}^n evidently changes the grading. It will be clear below that the general situation is from one side compatibility of all bilinear operations with the grading and from the other side with formal integration by parts. So, basic objects (local functionals etc.) are defined as equivalence classes modulo formal divergences (i.e., divergences of expressions containing θ -factors) and the unique decomposition into homogeneous subspaces with fixed grading function can be made only for representatives of these classes.

We call expressions of the form

$$\psi = \sum \int \theta^{(J)} D_K \psi_A^{(J)} \frac{\partial}{\partial \phi_A^{(K)}}$$

the *evolutionary vector fields*. Value of the evolutionary vector field on a local functional is given by the expression

$$\psi F = \sum \int \theta^{(I+J)} D_K \psi_A^{(J)} \frac{\partial f^{(I)}}{\partial \phi_A^{(K)}}.$$

In principle, this formula can be understood as a definition but we can interpret it also as a consequence of the standard relation

$$\frac{\partial \phi_A(y)}{\partial \phi_B(x)} = \delta(x, y) \delta_{AB}$$

and Rule 5.4 of I. It is a straightforward calculation to check that this operation is compatible with the formal integration by parts, i.e.

$$\psi \text{Div}(f) = \text{Div}(\psi f),$$

as it is in the standard formal variational calculus. This relation is, of course, valid for integrands.

It is easy to see that the evolutionary vector field with coefficients

$$\psi_A^{(J)} = \sum \left(D_L \xi_B^{(I)} \frac{\partial \eta_A^{(J-I)}}{\partial \phi_B^{(L)}} - D_L \eta_B^{(I)} \frac{\partial \xi_A^{(J-I)}}{\partial \phi_B^{(L)}} \right)$$

can be considered as the *commutator of the evolutionary vector fields* ξ and η

$$\psi F = [\xi, \eta] F = \xi(\eta F) - \eta(\xi F),$$

with the Jacobi identity fulfilled for the commutator operation, and so these vector fields form a Lie algebra.

3 Differentials and functional forms

The *differential of a local functional* is simply the first variation of it

$$dF = \sum \int \theta^{(J)} \frac{\partial f^{(J)}}{\partial \phi_A^{(K)}} \delta \phi_A^{(K)},$$

where here and below $\delta \phi_A^{(K)} = D_K \delta \phi_A$. It can also be expressed through the Fréchet derivative (Definition 2.13 of I) or through the higher Eulerian operators (Definition 2.4 of I)

$$dF = \sum \int \theta^{(J)} f^{(J)'}(\delta \phi) = \sum \int \theta^{(J)} D_K \left(E_A^K(f^{(J)}) \delta \phi_A \right).$$

This differential is a special example of functional 1-form. A general functional 1-form can be written as

$$\alpha = \sum \int \theta^{(J)} \alpha_{AK}^{(J)} \delta \phi_A^{(K)}.$$

Of course, the coefficients are not unique since we can make formal integration by parts. Let us call the following expression the *canonical form of a functional 1-form*

$$\alpha = \sum \int \theta^{(J)} \alpha_A^{(J)} \delta \phi_A.$$

Analogously, we can define *functional m-forms* as integrals or equivalence classes modulo formal divergences of vertical forms

$$\alpha = \frac{1}{m!} \sum \int \theta^{(J)} \alpha_{A_1 K_1, \dots, A_m K_m}^{(J)} \delta \phi_{A_1}^{(K_1)} \wedge \dots \wedge \delta \phi_{A_m}^{(K_m)}.$$

Define the *pairing* (or the *interior product*) of an evolutionary vector field and 1-form as

$$\alpha(\xi) = \xi \lrcorner \alpha = \sum \int \theta^{(I+J)} \alpha_{AK}^{(J)} D_K \xi_A^{(I)}. \quad (3.1)$$

The interior product of an evolutionary vector field and functional m -form will be

$$\begin{aligned} \xi \lrcorner \alpha &= \frac{1}{m!} \sum (-1)^{i+1} \int \theta^{(J+I)} \alpha_{A_1 K_1, \dots, A_m K_m}^{(J)} D_{K_i} \xi_{A_i}^{(I)} \delta \phi_{A_1}^{(K_1)} \wedge \dots \\ &\quad \dots \wedge \delta \phi_{A_{i-1}}^{(K_{i-1})} \wedge \delta \phi_{A_{i+1}}^{(K_{i+1})} \wedge \dots \wedge \delta \phi_{A_m}^{(K_m)}. \end{aligned}$$

Then the value of m -form on the m evolutionary vector fields will be defined by the formula

$$\alpha(\xi_1, \dots, \xi_m) = \xi_m \lrcorner \dots \xi_1 \lrcorner \alpha.$$

It can be checked by straightforward calculation that

$$\text{Div}(\alpha)(\xi_1, \dots, \xi_m) = \text{Div}(\alpha(\xi_1, \dots, \xi_m)).$$

The *differential of the m-form* given as

$$d\alpha = \frac{1}{m!} \sum \int \theta^{(J)} \frac{\partial \alpha_{A_1 K_1, \dots, A_m K_m}^{(J)}}{\partial \phi_A^{(K)}} \delta \phi_A^{(K)} \wedge \delta \phi_{A_1}^{(K_1)} \wedge \dots \wedge \delta \phi_{A_m}^{(K_m)},$$

satisfies standard properties

$$d^2 = 0$$

and

$$\begin{aligned} d\alpha(\xi_1, \dots, \xi_{m+1}) &= \sum_i (-1)^{i+1} \xi_i \alpha(\xi_1, \dots, \hat{\xi}_i, \dots, \xi_{m+1}) + \\ &\quad + \sum_{i < j} (-1)^{i+j} \alpha([\xi_i, \xi_j], \xi_1, \dots, \hat{\xi}_i, \dots, \hat{\xi}_j, \dots, \xi_{m+1}). \end{aligned}$$

The *Lie derivative* of a functional form α along an evolutionary vector field ξ can be introduced by the standard formula

$$L_\xi \alpha = \xi \lrcorner d\alpha + d(\xi \lrcorner \alpha).$$

4 Graded differential operators and their adjoints

We call linear matrix differential operators of the form

$$\hat{I} = \sum_{J \geq 0} \theta^{(J)} \sum_{N=0}^{N_{max}} I_{AB}^{(J)N} D_N$$

the *graded differential operators*.

Let us call the linear differential operator \hat{I}^* the *adjoint* to \hat{I} if for an arbitrary set of smooth functions f_A, g_A

$$\sum_{A,B} \int f_A \hat{I}_{AB} g_B = \sum_{A,B} \int g_A \hat{I}_{AB}^* f_B.$$

For coefficients of the adjoint operator we can derive the expression

$$I_{AB}^{*(J)M} = \sum_{K=0}^{K_{max}} \sum_{L=0}^{\min(K,J)} (-1)^{|K|} \binom{K}{L} \binom{K-L}{M} D_{K-L-M} I_{BA}^{(J-L)K}. \quad (4.1)$$

It is easy to check that the relation

$$\hat{I}(x) \delta(x, y) = \hat{I}^*(y) \delta(x, y)$$

follows from Rule 4.2 of I. For example, we have

$$\left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i} \right) \delta(x, y) = -\theta^{(i)} \delta(x, y). \quad (4.2)$$

In one of our previous publications [6] we tried to connect the appearance of surface terms in Poisson brackets and the standard manipulations with the δ -function. The ansatz used there for the above simplest example coincided with (4.2) up to the sign. The reason for the difference laid in the other choice made there instead of Rule 4.2 of I. That ansatz led us to the standard Poisson brackets which were not appropriate for boundary problems.

Operators satisfying relation

$$\hat{I}^* = -\hat{I}$$

will be called *skew-adjoint* ones. With the help of them it is possible to express 2-forms (and also 2-vectors to be defined below) in the canonical form

$$\alpha = \frac{1}{2} \sum_{A,B} \int \delta \phi_A \wedge \hat{I}_{AB} \delta \phi_B.$$

It is clear that we can consider representations of functional forms as decompositions over the basis derived as a result of the tensor product of $\delta \phi_A$, with the totally antisymmetric multilinear operators

$$\hat{\alpha} = \sum \theta^{(J)} \alpha_{A_1 K_1, \dots, A_m K_m}^{(J)} \left(D_{K_1} \cdot, \dots, D_{K_m} \cdot \right)$$

as coefficients of these decompositions.

5 Multi-vectors, mixed tensors and Schouten-Nijenhuis bracket

Let us introduce dual basis to $|\delta\phi_A\rangle$ by relation

$$\left\langle \frac{\delta}{\delta\phi_B(y)}, \delta\phi_A(x) \right\rangle = \delta_{AB}\delta(x, y) \quad (5.1)$$

and construct by means of the tensor product a basis

$$\frac{\delta}{\delta\phi_{B_1}(y)} \otimes \frac{\delta}{\delta\phi_{B_2}(y)} \otimes \dots \otimes \frac{\delta}{\delta\phi_{B_m}(y)}.$$

Then by using totally antisymmetric multilinear operators described in previous Section we can define *functional m-vectors* (or *multi-vectors*)

$$\psi = \frac{1}{m!} \sum \int \theta^{(J)} \psi_{B_1 L_1, \dots, B_m L_m}^{(J)} D_{L_1} \frac{\delta}{\delta\phi_{B_1}} \wedge \dots \wedge D_{L_m} \frac{\delta}{\delta\phi_{B_m}}.$$

Here a natural question on the relation between evolutionary vector fields and 1-vectors arises. Evidently, evolutionary vector fields lose their form when being integrated by parts whereas 1-vectors conserve it. Let us make a partial integration in the expression of a general evolutionary vector field

$$\xi = \sum \int \theta^{(J)} D_K \xi_A^{(J)} \frac{\partial}{\partial\phi_A^{(K)}}$$

by removing D_K from $\xi_A^{(J)}$, then we get

$$\xi = \sum \int \xi_A^{(J)} \theta^{(J+L)} (-1)^{|K|} \binom{K}{L} D_{K-L} \frac{\partial}{\partial\phi_A^{(K)}}.$$

It is easy to see that by using Rule 5.4 from I in the backward direction we can write

$$\xi = \sum \int [\theta^{(J)} \xi_A^{(J)}] [\theta^{(L)} (-1)^{|L|} E_A^L] = \sum \int \theta^{(J)} \xi_A^{(J)} \frac{\delta}{\delta\phi_A},$$

where the higher Eulerian operators and full variational derivative (Definition 5.1 of I) are consequently used. Therefore, we have proved a following Statement.

Statement 5.1 *There is a one-to-one correspondence between evolutionary vector fields and functional 1-vectors. The coefficients of 1-vector in the canonical form $\xi_A^{(J)}$ are equal to the characteristic of the evolutionary vector field.*

It is not difficult to show that we can deduce the pairing (interior product) of 1-forms and 1-vectors and this pairing preserves this identification. Really, the definition of the dual basis (5.1) and Rules 4.2, 5.4 of I permit us to derive that

$$\alpha(\xi) = \xi \lrcorner \alpha = \sum \int \int \theta^{(I)}(x) \theta^{(J)}(y) \alpha_{AK}^{(I)}(x) \xi_{BL}^{(J)}(y) \left\langle D_L \frac{\delta}{\delta\phi_B(y)}, D_K \delta\phi_A(x) \right\rangle =$$

$$= \sum \int \theta^{(I+J)} D_L \alpha_{AK}^{(I)} D_K \xi_{AL}^{(J)} = \sum \int \theta^{(I+J)} \text{Tr}(\alpha^{(I)} \xi^{(J)}),$$

and when 1-vector is in the canonical form (only $L = 0$ term is nonzero) this result coincides with Eq.(3.1).

This formula for the pairing will be exploited below also for interior product of 1-vectors and m -forms or 1-forms and m -vectors. Its importance comes from the fact that it is invariant under the formal partial integration both in forms and in vectors, i.e.,

$$\text{Div}(\alpha)(\xi) = \text{Div}(\alpha(\xi)) = \alpha(\text{Div}(\xi)).$$

Evidently, it is the trace construction for convolution of differential operators (as coefficients of tensor objects in the proposed basis) that guarantees this invariance.

The interior product of 1-vector onto m -form and, analogously, of 1-form onto m -vector is defined as

$$\begin{aligned} \xi \lrcorner \alpha &= \frac{1}{m!} \sum (-1)^{(i+1)} \int \theta^{(I+J)} D_{K_i} \xi_{A_i L}^{(I)} D_L \left(\alpha_{A_1 K_1, \dots, A_m K_m}^{(J)} \delta \phi_{A_1}^{(K_1)} \wedge \dots \right. \\ &\quad \left. \dots \wedge \delta \phi_{A_{i-1}}^{(K_{i-1})} \wedge \delta \phi_{A_{i+1}}^{(K_{i+1})} \wedge \dots \wedge \delta \phi_{A_m}^{(K_m)} \right). \end{aligned}$$

Then we also can define the value of m -form on m 1-vectors (or, analogously, m -vector on m 1-forms)

$$\alpha(\xi_1, \dots, \xi_m) = \xi_m \lrcorner \dots \xi_1 \lrcorner \alpha = \sum \int \theta^{(J+I_1+\dots+I_m)} \text{Tr} \left(\alpha^{(J)} \xi_1^{(I_1)} \dots \xi_m^{(I_m)} \right),$$

where each entry of multilinear operator α acts only to the one ξ , whereas each derivation of the operator ξ acts on the product of α and all the rest of ξ 's.

It is possible to define the *differential of m -vector*

$$d\psi = \frac{1}{m!} \sum \int \theta^{(J)} \frac{\partial \psi_{A_1 K_1, \dots, A_m K_m}^{(J)}}{\partial \phi_B^{(L)}} \delta \phi_B^{(L)} D_{K_1} \frac{\delta}{\delta \phi_{A_1}} \wedge \dots \wedge D_{K_m} \frac{\delta}{\delta \phi_{A_m}},$$

as an example of a mixed $\binom{m}{1}$ object. Evidently, $d^2\psi = 0$.

With the help of the previous constructions we can define the *Schouten-Nijenhuis bracket*

$$[\xi, \eta]_{SN} = d\xi \lrcorner \eta + (-1)^{pq} d\eta \lrcorner \xi$$

for two multi-vectors of orders p and q . The result of this operation is $p + q - 1$ -vector and it is analogous to the Schouten-Nijenhuis bracket in tensor analysis [4]. Its use in the formal variational calculus is described in Refs.[2],[3]. However, in cited references this bracket is usually defined for operators. We can recommend Ref.[7] as an interesting source for the treatment of the Schouten-Nijenhuis bracket of multi-vectors. Our construction of this bracket guarantees a compatibility with the equivalence modulo divergences

$$[\text{Div}(\xi), \eta]_{SN} = \text{Div}[\xi, \eta]_{SN} = [\xi, \text{Div}(\eta)]_{SN}.$$

Statement 5.2 *The Schouten-Nijenhuis bracket of functional 1-vectors up to a sign coincides*

with the commutator of the corresponding evolutionary vector fields.

Proof. Let us take the two 1-vectors in canonical form without loss of generality

$$\xi = \sum \int \theta^{(J)} \xi_A^{(J)} \frac{\delta}{\delta \phi_A}, \quad \eta = \sum \int \theta^{(K)} \eta_B^{(K)} \frac{\delta}{\delta \phi_B}$$

and compute

$$[\xi, \eta]_{SN} = d\xi \lrcorner \eta - d\eta \lrcorner \xi.$$

We have

$$d\xi = \sum \int \theta^{(J)} \xi_A^{(J)'} (\delta \phi) \frac{\delta}{\delta \phi_A} = \sum \int \theta^{(J)} \frac{\partial \xi_A^{(J)}}{\partial \phi_C^{(L)}} \delta \phi_C^{(L)} \frac{\delta}{\delta \phi_A},$$

and

$$d\xi \lrcorner \eta = - \sum \int \theta^{(J+K)} \frac{\partial \xi_A^{(J)}}{\partial \phi_B^{(L)}} D_L \eta_B^{(K)} \frac{\delta}{\delta \phi_A}.$$

Therefore, we obtain

$$[\xi, \eta]_{SN} = - \sum \int \theta^{(J+K)} \left(D_L \eta_B^{(K)} \frac{\partial \xi_A^{(J)}}{\partial \phi_B^{(L)}} - D_L \xi_B^{(K)} \frac{\partial \eta_A^{(J)}}{\partial \phi_B^{(L)}} \right) \frac{\delta}{\delta \phi_A} = -[\xi, \eta],$$

and the proof is completed.

Statement 5.3 (Olver's Lemma [2]) *The Schouten-Nijenhuis bracket for two bivectors can be expressed in the form*

$$[\xi, \psi]_{SN} = -\frac{1}{2} \sum \int \xi \wedge \hat{I}'(\hat{K}\xi) \wedge \xi - \frac{1}{2} \sum \int \xi \wedge \hat{K}'(\hat{I}\xi) \wedge \xi, \quad (5.2)$$

where the two differential operators \hat{I} , \hat{K} are the coefficients of the bivectors in their canonical form.

Proof. Let us consider the Schouten-Nijenhuis bracket for the two bivectors and without loss of generality take them in the canonical form

$$\chi = \frac{1}{2} \sum \int \theta^{(L)} \xi_A \wedge I_{AB}^{(L)N} D_N \xi_B,$$

$$\psi = \frac{1}{2} \sum \int \theta^{(M)} \xi_C \wedge K_{CD}^{(M)P} D_P \xi_D,$$

where $\xi_A = \delta/\delta \phi_A$ and operators \hat{I} , \hat{K} are skew-adjoint. Then we have

$$d\chi = \frac{1}{2} \sum \int \theta^{(L)} \frac{\partial I_{AB}^{(L)N}}{\partial \phi_E^{(J)}} \delta \phi_E^{(J)} \xi_A \wedge D_N \xi_B$$

and

$$d\chi \lrcorner \psi = \frac{1}{4} \sum \int \theta^{(L+M)} \frac{\partial I_{AB}^{(L)N}}{\partial \phi_C^{(J)}} D_J \left(K_{CD}^{(M)P} D_P \xi_D \right) \wedge \xi_A \wedge D_N \xi_B -$$

$$-\frac{1}{4} \sum \int \theta^{(L+M)} D_P \left(\frac{\partial I_{AB}^{(L)N}}{\partial \phi_D^{(J)}} \xi_A \wedge D_N \xi_B \right) \wedge D_J (\xi_C K_{CD}^{(M)P}).$$

Now let us make integration by parts in the second term

$$\begin{aligned} d\chi \lrcorner \psi &= -\frac{1}{4} \sum \int \theta^{(L+M)} \xi_A \wedge (I_{AB}^{(L)N})' \left(\hat{K}^{(M)} \xi \right) \wedge D_N \xi_B - \\ &-\frac{1}{4} \sum \int \theta^{(L+M+Q)} (-1)^{|P|} \binom{P}{Q} \frac{\partial I_{AB}^{(L)N}}{\partial \phi_D^{(J)}} \xi_A \wedge D_N \xi_B \wedge D_{J+P-Q} (\xi_C K_{CD}^{(M)P}). \end{aligned}$$

At last we change the order of multipliers under wedge product in the second term, make a replacement $M \rightarrow M - Q$ and organize the whole expression in the form

$$\begin{aligned} d\chi \lrcorner \psi &= -\frac{1}{4} \sum \int \theta^{(L+M)} \xi_A \wedge (I_{AB}^{(L)N})'_C \left(\hat{K}_{CD}^{(M)} \xi_D - \right. \\ &\left. - (-1)^{|P|} \binom{P}{Q} \binom{P-Q}{R} D_{P-Q-R} K_{CD}^{(M-Q)P} D_R \xi_C \right) \wedge D_N \xi_B. \end{aligned}$$

Having in mind the definition of adjoint operator (4.1) we can represent the final result of the calculation as follows,

$$[\xi, \psi]_{SN} = -\frac{1}{2} \sum \int \theta^{(L+M)} \xi \wedge \left((\hat{I}^{(L)})' (\hat{K}^{(M)} \xi) - (\hat{K}^{(M)})' (\hat{I}^{(L)} \xi) \right) \wedge \xi,$$

thus supporting in this extended formulation the method, proposed in Ref. [2] for testing the Jacobi identity (see Section 7).

6 Poisson brackets and Hamiltonian vector fields

Let us call a bivector

$$\Psi = \frac{1}{2} \sum \int \frac{\delta}{\delta \phi_A} \wedge \hat{I}_{AB} \frac{\delta}{\delta \phi_B},$$

formed with the help of the graded skew-adjoint differential operator

$$\hat{I}_{AB} = \sum \theta^{(L)} I_{AB}^{(L)N} D_N,$$

the *Poisson bivector* if

$$[\Psi, \Psi]_{SN} = 0.$$

The operator \hat{I}_{AB} is then called the *Hamiltonian operator*. We call the value of the Poisson bivector on the differentials of two functionals F and G

$$\{F, G\} = \Psi(dF, dG) = dG \lrcorner dF \lrcorner \Psi$$

the *Poisson bracket* of these functionals.

The explicit form of the Poisson brackets can easily be obtained. It depends on the explicit form of the functional differential, which can be changed by partial integration. Of

course, all the possible forms are equivalent. Taking the extreme cases we get an expression through Fréchet derivatives

$$\{F, G\} = \sum \int \theta^{(J)} \text{Tr} \left(f'_A \hat{I}_{AB}^{(J)} g'_B \right) \quad (6.1)$$

or through higher Eulerian operators

$$\{F, G\} = \sum \int \theta^{(J)} D_{P+Q} \left(E_A^P(f) \hat{I}_{AB}^{(J)} E_B^Q(g) \right). \quad (6.2)$$

Theorem 6.1 *The Poisson bracket defined above satisfies Definition 2.3 of I.*

Proof. Equivalence of these definitions follows from the three facts: 1) from the previous formulas (6.1), (6.2) it is clear that $\{F, G\}$ is a local functional, 2) antisymmetry of $\{F, G\}$ is evident and 3) equivalence of the Jacobi identity to the Poisson bivector property (to be proved in Section 7).

The result of interior product of the differential of a local functional H on the Poisson bivector (up to the sign) will be called the *Hamiltonian vector field* (or the *Hamiltonian 1-vector*)

$$\hat{I}dH = -dH \lrcorner \Psi$$

corresponding to the Hamiltonian H . Evidently, the standard relations take place

$$\{F, H\} = dF(\hat{I}dH) = (\hat{I}dH)F.$$

Theorem 6.2 *The Hamiltonian vector field corresponding to the Poisson bracket of the functionals F and H coincides up to the sign with commutator of the Hamiltonian vector fields corresponding to these functionals.*

Proof. Consider a value of the commutator of Hamiltonian vector fields $\hat{I}dF$ and $\hat{I}dH$ on the arbitrary functional G

$$\begin{aligned} [\hat{I}dF, \hat{I}dH]G &= \hat{I}dF(\hat{I}dH(G)) - \hat{I}dH(\hat{I}dF(G)) = \\ &= \hat{I}dF(\{G, H\}) - \hat{I}dH(\{G, F\}) = \{\{G, H\}, F\} - \{\{G, F\}, H\} = \\ &= -\{G, \{F, H\}\} = -\hat{I}d\{F, H\}(G), \end{aligned}$$

where we have used the Jacobi identity and antisymmetry of Poisson bracket. Due to the arbitrariness of G the proof is completed.

Example 6.3

Let us consider a first structure

$$\{u(x), u(y)\} = \frac{1}{2}(D_x - D_y)\delta(x, y)$$

of the Korteweg-de Vries equation (Example 7.6 of Ref. [2])

$$u_t = u_{xxx} + uu_x.$$

Construct the adjoint graded operator to θD according to Eq.(4.1)

$$(\theta D)^* = -\theta D - D\theta$$

and the skew-adjoint operator is

$$\hat{I} = \frac{1}{2} \left(\theta D - (\theta D)^* \right) = \theta D + \frac{1}{2} D\theta.$$

The Poisson bivector has a form

$$\Psi = \frac{1}{2} \int \theta \left(\frac{\delta}{\delta u} \wedge D \frac{\delta}{\delta u} \right).$$

The differential of a local functional H (for simplicity it is written in canonical

$$H = \int \theta h$$

form) is equal to

$$dH = \int \theta h'(\delta u) = \sum_{k=0}^{\infty} \int \theta^{(k)} (-1)^k E^k(h) \delta u,$$

where the Fréchet derivative or higher Eulerian operators can be used. Therefore, the Hamiltonian vector field generated by H is

$$\hat{I} dH = -dH \lrcorner \Psi = -\frac{1}{2} \int \theta \left[h' \left(D \frac{\delta}{\delta u} \right) - D h' \left(\frac{\delta}{\delta u} \right) \right],$$

or

$$-\frac{1}{2} \int \theta^{(k)} (-1)^k \left[E^k(h) D - D E^k(h) \right] \frac{\delta}{\delta u},$$

or also

$$-\frac{1}{2} \int \theta^{(k)} (-1)^k D_i \left[E^k(h) D - D E^k(h) \right] \frac{\partial}{\partial u^{(i)}}.$$

The value of this vector field on another functional F coincides with the Poisson bracket

$$-dF \lrcorner dH \lrcorner \Psi = \{F, H\} = \frac{1}{2} \sum \int \theta^{(k+l)} (-1)^{k+l} \left(E^k(f) D E^l(h) - E^k(h) D E^l(f) \right).$$

7 Proof of Jacobi identity

In this section we will prove that the Jacobi identity for the Poisson bracket is fulfilled if and only if the Schouten–Nijenhuis bracket of the corresponding Poisson bivector with itself is equal to zero. This should complete the proof of Theorem 6.1.

Let us use one of the possible forms of the Poisson brackets given in Appendix of I

$$\{F, G\} = \frac{1}{2} \sum \int \theta^{(J)} \text{Tr} \left(f'(\hat{I}^{(J)} g') - g'(\hat{I}^{(J)} f') \right),$$

where the differential operator \hat{I} is not supposed to be skew-adjoint for the easier comparison of this proof with that given in I. We remind the reader that in less condensed notations

$$\text{Tr}\left(f'(\hat{I}g')\right) = \sum \binom{J}{M} \binom{K}{L} D_L \frac{\partial f}{\partial \phi_A^{(J)}} D_{J+K-L-M} I_{AB}^N D_{N+M} \frac{\partial g}{\partial \phi_B^{(K)}}$$

(in Appendix of I the indices M and L in the binomial coefficients of the same formula are unfortunately given in the opposite order).

We will estimate the bracket

$$\{\{F, G\}, H\} = \frac{1}{2} \sum \int \theta^{(J)} \text{Tr} \left[\{f, g\}'(\hat{I}^{(J)} h') - h'(\hat{I}^{(J)} \{f, g\}') \right],$$

where $\{f, g\}$ denotes the integrand of $\{F, G\}$. Since Fréchet derivative is a derivation we have

$$\{f, g\}' = \frac{1}{2} \sum \theta^{(K)} \text{Tr} \left(f''(\hat{I}^{(K)} g', \cdot) + f' \hat{I}^{(K)}(\cdot) g' + g''(f' \hat{I}^{(K)}, \cdot) - (f \leftrightarrow g) \right)$$

and

$$\text{Tr} \left[\{f, g\}' \hat{I} h' \right] = \frac{1}{2} \left[f''(\hat{I} g', \hat{I} h') + f' \hat{I}(\hat{I} h') g' + g''(f' \hat{I}, \hat{I} h') - (f \leftrightarrow g) \right].$$

Let us explain that f'' denotes the second Fréchet derivative, i.e., the symmetric bilinear operator arising in calculation of the second variation of the local functional F (in the canonical form):

$$f''(\xi, \eta) = \sum_{A,B} \sum_{J,K} \frac{\partial^2 f}{\partial \phi_A^{(J)} \partial \phi_B^{(K)}} D_J \xi_A D_K \eta_B.$$

When we put into entries of f'' operators under the trace sign it should be understood that these operators act on everything except their own coefficients, for example,

$$\begin{aligned} \text{Tr} \left(f''(\hat{I} g', \hat{I} h') \right) &= \sum \binom{L}{P} \binom{L-P}{Q} \binom{M}{S} \binom{M-S}{T} \times \\ &\times D_{L+M-P-Q-S-T} \frac{\partial^2 f}{\partial \phi_A^{(J)} \partial \phi_B^{(K)}} D_{J+T} \left(D_P \hat{I}_{AC} \frac{\partial g}{\partial \phi_C^{(L)}} \right) D_{K+Q} \left(D_S \hat{I}_{BD} \frac{\partial h}{\partial \phi_D^{(M)}} \right) \end{aligned}$$

and the expression remains symmetric under permutation of its entries

$$\text{Tr} \left(f''(\hat{I} g', \hat{I} h') \right) = \text{Tr} \left(f''(\hat{I} h', \hat{I} g') \right).$$

When the operator \hat{I} stands to the right from the operator of Fréchet derivative f' as in expression

$$\text{Tr} \left(g''(\hat{I} h', f' \hat{I}) \right),$$

it acts on everything except f' . At last, for Fréchet derivative of the operator we have

$$\hat{I}'(\hat{I} h') = \sum \frac{\partial I_{AB}^K}{\partial \phi_C^{(J)}} D_J \left(I_{CD}^L D_L \frac{\partial h}{\partial \phi_D^{(M)}} D_M \right) D_K.$$

Making similar calculations we get

$$\text{Tr} \left[h' \hat{I} \{f, g\}' \right] = \frac{1}{2} \text{Tr} \left(f''(h' \hat{I}, \hat{I} g') + f' \hat{I}'(h' \hat{I}) g' + g''(f' \hat{I}, h' \hat{I}) - (f \leftrightarrow g) \right)$$

and therefore

$$\begin{aligned} \{\{F, G\}, H\} &= \frac{1}{4} \sum \int \theta^{(J+K)} \text{Tr} \left(f''(\hat{I}^{(J)} g', \hat{I}^{(K)} h') - f''(h' \hat{I}^{(J)}, \hat{I}^{(K)} g') - \right. \\ &\quad \left. - f''(\hat{I}^{(J)} h', g' \hat{I}^{(K)}) + f''(g' \hat{I}^{(J)}, h' \hat{I}^{(K)}) + f' \hat{I}'^{(J)}(\hat{I}^{(K)} h' - h' \hat{I}^{(K)}) g' - (f \leftrightarrow g) \right). \end{aligned}$$

Just the first four terms, apart from the fifth containing Fréchet derivative of the operator \hat{I} , were present in our proof for nonultralocal case given in I (only terms with zero grading were allowed for \hat{I} there). After cyclic permutation of F, G, H all terms with the symmetric operator of the second Fréchet derivative are mutually cancelled and

$$\begin{aligned} \{\{F, G\}, H\} + \text{c.p.} &= \frac{1}{4} \int \theta^{(J+K)} \text{Tr} \left(f' \hat{I}'^{(J)}(\hat{I}^{(K)} h' - h' \hat{I}^{(K)}) g' - \right. \\ &\quad \left. - g' \hat{I}'^{(J)}(\hat{I}^{(K)} h' - h' \hat{I}^{(K)}) f' + \text{c.p.} \right), \end{aligned}$$

where cyclic permutations of F, G , and H are abbreviated to *c.p.*. When operator \hat{I} is given in explicitly skew-adjoint form all the four terms are equal. Taking into account Olver's Lemma (5.2) we get

$$\{\{F, G\}, H\} + \text{c.p.} = -[\hat{I}, \hat{I}]_{SN}(\text{d}F, \text{d}G, \text{d}H),$$

so finishing the proof.

8 Examples of nonultralocal operators

The second structure of the Korteweg-de Vries equation may serve as a counter-example to hypothesis [5] that all operators which are Hamiltonian with respect to the standard Poisson brackets should also be Hamiltonian in the new brackets.

Example 8.1

Let us start with the standard expression (Example 7.6 of Ref. [2])

$$\{u(x), u(y)\} = \left(\frac{d^3}{dx^3} + \frac{2}{3} u \frac{d}{dx} + \frac{1}{3} \frac{du}{dx} \right) \delta(x, y)$$

and construct the adjoint operator to

$$\hat{K} = \theta(D_3 + \frac{2}{3}uD + \frac{1}{3}Du),$$

which is

$$\hat{K}^* = -\theta(D_3 + \frac{2}{3}uD + \frac{1}{3}Du) - D\theta(3D_2 + \frac{2}{3}u) - 3D_2\theta D - D_3\theta.$$

Then the skew-adjoint operator

$$\hat{I} = \frac{1}{2}(\hat{K} - \hat{K}^*) = \theta(D_3 + \frac{2}{3}uD + \frac{1}{3}Du) + D\theta(\frac{3}{2}D_2 + \frac{1}{3}u) + \frac{3}{2}D_2\theta D + \frac{1}{2}D_3\theta$$

can be used for forming the bivector

$$\Psi = \frac{1}{2} \int \xi \wedge \hat{I}\xi,$$

where $\delta/\delta u = \xi$. This bivector has a form

$$\Psi = \frac{1}{2} \int \left(\theta\xi \wedge D_3\xi + \frac{3}{2}D\theta\xi \wedge D_2\xi + \left(\frac{3}{2}D_2\theta + \frac{2}{3}\theta u\right)\xi \wedge D\xi \right).$$

Then evaluating the Schouten-Nijenhuis bracket for the bivector with the help of Statement 5.3

$$[\Psi, \Psi]_{SN} = \int \left(\frac{2}{3}\theta\xi \wedge D_3\xi \wedge D\xi + D\theta\xi \wedge D_2\xi \wedge D\xi \right)$$

and integrating the first term by parts we get

$$[\Psi, \Psi]_{SN} = \frac{1}{3} \int \theta D(\xi \wedge D\xi \wedge D_2\xi).$$

Therefore, instead of the Jacobi identity we have

$$\{\{F, G\}, H\} + c.p. = -\frac{1}{3} \sum_{i,j,k=0}^{\infty} \int_{\Omega} D_{i+j+k+1} \left(E^i(f) D E^j(g) D_2 E^k(h) + c.p. \right) dx.$$

So, the second structure of KdV equation can be Hamiltonian only under special boundary conditions.

Example 8.2

Now let consider another example which is also nonultralocal, but the operator remains to be Hamiltonian in the new brackets independently of boundary conditions. The Euler equations for the flow of ideal fluid can be written [2] in Hamiltonian form as follows (Example 7.10 of Ref. [2])

$$\frac{\partial \omega}{\partial t} = \mathcal{D} \frac{\delta H}{\delta \omega},$$

where

$$H = \int \frac{1}{2} |\mathbf{u}|^2 dx, \quad \omega = \nabla \times \mathbf{u}.$$

Let us limit our consideration by the 2-dimensional case when ω has only one component ω and

$$\mathcal{D} = \omega_x D_y - \omega_y D_x,$$

where $\omega_i = D_i \omega$, $i = (x, y)$. We can construct the skew-adjoint operator

$$\hat{I} = \frac{1}{2} \left(\theta \mathcal{D} - (\theta \mathcal{D})^* \right) = \theta(\omega_x D_y - \omega_y D_x) + \frac{1}{2} (D_y \theta \omega_x - D_x \theta \omega_y),$$

and then the bivector

$$\Psi = \frac{1}{2} \int \xi \wedge \hat{I}\xi = \frac{1}{2} \int \theta(\omega_x \xi \wedge \xi_y - \omega_y \xi \wedge \xi_x),$$

where $\xi = \delta/\delta\omega$. Statement 5.3 gives us

$$\begin{aligned} [\Psi, \Psi]_{SN} = \int & \left(\theta \left[\omega_x (\xi \wedge \xi_{xy} \wedge \xi_y - \xi \wedge \xi_{yy} \wedge \xi_x) + \omega_y (\xi \wedge \xi_{xy} \wedge \xi_x - \xi \wedge \xi_{xx} \wedge \xi_y) \right] + \right. \\ & \left. + \left[D_y \theta \omega_x - D_x \theta \omega_y \right] \xi \wedge \xi_x \wedge \xi_y \right) \end{aligned}$$

and after integration by parts the expression can be reduced to zero.

9 Conclusion

We have shown that there is an extension of the standard formal variational calculus which incorporates the real divergences without any specification of the boundary conditions on the boundary of a finite domain. It would be important to understand relations of this formalism to the constructions of the variational bicomplex [8]. It seems also rather interesting to study if some physically relevant algebras can be realized with the help of the new Poisson brackets as algebras of local functionals. It is not clear for us now whether the Hamiltonian equations generated by the new brackets can be solved in some space of functions and what kind of space could be used for this purpose.

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